Annex No. 10 to the MU Directive on Habilitation Procedures and Professor Appointment Procedures

HABILITATION THESIS REVIEWER'S REPORT

Masaryk University	
Applicant	Phuoc Tai Nguyen
Habilitation thesis	Boundary value problems for nonlinear elliptic equations with a Hardy potential
Reviewer Reviewer's	Prof. Louis Dupaigne
home unit, institution	Université Claude Bernard Lyon 1

The review text is attached below.

Conclusion

The habilitation thesis entitled “Boundary value problems for nonlinear elliptic equations with a Hardy potential” by Phuoc Tai Nguyen **fulfils** requirements expected of a habilitation thesis in the field of Mathematics – Mathematical Analysis.

Date: August 16, 2021 Signature:

Report on the manuscript
Boundary value problems for nonlinear elliptic equations with a Hardy potential
presented by Phuoc-Thai Nguyen

referee: Louis Dupaigne

A classical result in potential theory, the Herglotz theorem, asserts that positive harmonic functions u defined over a bounded Lipschitz domain of Euclidean space $\Omega \subset \mathbb{R}^N$, $N \geq 1$, are in one-to-one correspondance with their boundary traces, the positive bounded Radon measures ν over $\partial\Omega$, through the Poisson formula

$$u(x) = \int_{\partial\Omega} P(x,y)d\nu(y), \qquad x \in \Omega. \tag{1}$$

The fact that all positive solutions of the model nonlinear elliptic PDE

$$\Delta u = u^2 \qquad \text{in } \Omega \tag{2}$$

could be similarly described by their boundary values is a major breakthrough due to the impetus of Le Gall and the following works of Dhersin, Mselati, as well as Dynkin and Kuznetsov. The theory was further extended to more general nonlinearities such as u^q, $q > 1$, by probabilistic and analytic means notably by Dynkin, and by Marcus and Véron through analytic methods.

If one perturbs the Laplace operator by a Hardy potential, namely if one considers the operator

$$L_\mu = \Delta + \frac{\mu}{\delta^2},$$

where $\mu \in \mathbb{R}$ is chosen so that the operator is coercive and where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, the picture changes significantly, even in the linear setting. A representation formula of the type (1) still holds, where the Poisson kernel is to be replaced with the Martin kernel of the operator, as follows from Ancona's work. But the standard notion of trace, obtained by averaging the values of the function over a smooth approximation of the boundary and passing to the limit, is meaningless in this setting. Indeed, due to the singularity of the potential, any positive L_μ-harmonic function must vanish *in average* on the boundary.

A first nice concept of this manuscript, introduced by NGuyen with Marcus, is the definition of normalized trace[1] of a function: a function u has trace ν in this sense if it agrees with the right-hand-side of (1) after averaging over the approximation of the boundary $[\delta(x) = \beta]$, normalizing by the small factor $\beta^{1-\alpha}$, where $\alpha < 1$ is the first indicial root of the operator L_μ, and letting $\beta \to 0$. In particular, one recovers a full one-to-one correspondance between positive harmonic functions and their traces in this setting. As demonstrated by NGuyen and his coauthors, this is also the proper way to generalize the standard linear theory of the Laplace operator in the measure framework to the operator L_μ, with delicate adaptations of the function and test function spaces. In this direction, let me mention that he derived with Gkikas the nontrivial boundary regularity theory (in particular the pointwise and gradient bounds in the appropriate weighted weak-Lebesgue spaces, naturally generalizing the spirit of Section 5.3 of Ponce's book "Elliptic PDEs, Measures and Capacities"), based on earlier knowledge of sharp heat kernel bounds due to Filippas, Moschini and Tertikas and a layer-cake-type decomposition lemma due to Bidaut-Véron and Vivier. Another nice result, demonstrated by NGuyen with Gkikas, states that the notion of trace he introduced agrees with another notion of trace due to Gkikas and Véron, obtained through weak convergence and harmonic measures defined on interior approximations of the domain. Thanks to earlier work of Gkikas and Véron, this gives in particular a pointwise understanding of the normalized trace when the datum is continuous: for example, if $\mu < 1/4$ and $h \in C(\partial\Omega)$, there exists a unique harmonic function u such that

$$\lim_{x\in\Omega, x\to y} \frac{u(x)}{\delta(x)^{1-\alpha}} = h(y), \qquad y \in \partial\Omega.$$

[1] This bears interesting resemblances with the boundary-value problem for the fractional Laplacian, see in particular the work of N. Abatangelo.

I want to underline that the efforts of the author to unify the theory parallel the quality of the writing of his memoir. This is not so often the case and is much appreciated.

Let us turn to the nonlinear theory. Already for the model equation (2), one should refrain from thinking that any boundary data is admissible for the function u. Indeed, in dimension $N \geq 3$, the problem becomes supercritical: as originally proved by Gmira and Véron, isolated boundary singularities are removable and boundary traces of the form $\nu = \delta_y$, $y \in \partial\Omega$ are ruled out. In addition, the boundary value $+\infty$ is also allowed, thanks to the classical works of Keller and Osserman, so that the class of bounded Radon measures over $\partial\Omega$ is not large enough to capture all solutions of the equation. In order to classify all solutions in terms of traces, one is lead to first consider the case of so-called moderate solutions, that is, those positive solutions which can be bounded above by a positive harmonic function. Dynkin and Kuznetsov showed that there is a one-to-one correspondance between moderate solutions of (2) and bounded Radon measures ν on the boundary $\partial\Omega$ which do not charge sets of 0 boundary capacity. Mselati, building on the work of Le Gall, then completed the classification, by showing that every positive solution of (2) is the increasing limit of moderate solutions, and can be represented in terms of its so-called fine trace. Roughly speaking, this trace is given by a subset of the boundary where the solution blows up and a measure on its complement, to which the solution converges non-tangentially.

For the operator L_μ and $q > 1$, the model problem in case of nonlinear absorption now reads

$$L_\mu u = u^q \qquad \text{in } \Omega$$

Together with Marcus, NGuyen naturally considered the class of positive moderate solutions. They proved that these are the solutions having as normalized trace a positive finite Radon measure ν. For each such measure, there exists at most one solution, which depends monotonically on ν and a natural weighted-L^1 estimate holds. In addition, the ratio of $u(x)$ over the right-hand side of (1) converges as x approaches the boundary non-tangentially. They also identified the critical exponent for the unconditional existence of a moderate solution. Precisely, they showed that in the subcritical regime, for every positive finite Radon measure ν, there exists a unique moderate solution with normalized trace ν, while in the supercritical case, no solution exists if ν is a Dirac mass δ_y, $y \in \partial\Omega$ or a positive multiple thereof. In fact, as proved later by Gkikas and Véron, the existence of moderate solutions in the supercritical case is completely characterized in terms of adequate capacities, in which paper the full classification of positive solutions in the subcritical regime is also established. Such a classification seems still partially open in the supercritical case: could the author confirm this point ?

This being asked, let us look at further generalizations of the problem, namely gradient dependant nonlinearities. NGuyen considered quite general equations of the type

$$L_\mu u = g(u, |\nabla u|) \qquad \text{in } \Omega, \tag{3}$$

where g is any nonnegative function which is nondecreasing and locally Lipschitz in both variables and vanishes at $(0,0)$. The additive case $g(u, |\nabla u|) = u^p + |\nabla u|^q$ and the multiplicative case $g(u, |\nabla u|) = u^p|\nabla u|^q$ serve as emblematic models. A first feat of Nguyen with Gkikas is the identification of the subcritical regime at this level of generality, which takes the form of an integral criterion on g (in the additive case, subcritical nonlinearities are simply those for which both p and q are subcritical). More precisely, for all subcritical nonlinearities, he proves the existence of moderate solutions with arbitrary normalized trace $\nu \in \mathfrak{M}^+(\partial\Omega)$. This is done by first truncating the nonlinearity and approximating the problem on an interior subdomain, where the singularity is no longer present and the Perron-Remak method is successfully applied through a delicate use of Schauder's fixed point theorem. Passing to the limit in the domain is easy thanks to the linear theory. Passing to the limit in the nonlinearity is quite more delicate. I mention in passing that extending this existence result to nonlinearities g with no monotonicity property could be an interesting challenge.

Uniqueness of moderate solutions turns out to be also a subtle affair and only the model nonlinearities are treated. In the multiplicative case, this is done through a comparison lemma, which exploits a gradient bound for subsolutions of the Dirichlet problem (thanks to the additional regularity enjoyed by superharmonic functions) due to the authors and a very clever use of Kato's inequality in integral form, a strategy which was already successfully conducted by NGuyen alone in the simpler additive case, inspired by earlier work of Porretta in a different setting.

Turning to non-moderate solutions, the first examples occur when dealing with isolated singularities. For the additive (with Marcus) and multiplicative nonlinearities (with Marcus and then with Gkikas) in the subcritical regime, he constructs a singular solution u_∞ with boundary value $\infty\delta_0$, where $0 \in \partial\Omega$, that is, u_∞ is the pointwise increasing limit of the moderate solutions with boundary value $k\delta_0$, as $k \to +\infty$. He also gives a complete description of isolated singularities. In particular, the constructed non-moderate solution can be blown down onto a homogeneous function $|x|^{-\gamma}w(x/|x|)$, where $\gamma > 0$ can be explicitly computed in terms of the nonlinearity and w solves the expected nonlinear elliptic equation (with Dirichlet condition) on the half-sphere.

For the additive and multiplicative nonlinearities in the supercritical regime, NGuyen obtains a necessary condition (in terms of adequate capacities again) for the existence of moderate solutions, as well as a removable singularity result for the same capacities. These results are natural and important steps towards the full classification of positive solutions.

The last part of the manuscript is dedicated to equations with source nonlinearity i.e. equations of the form

$$-L_\mu u = g(u, |\nabla u|) \qquad \text{in } \Omega, \tag{4}$$

where $g \geq 0$ depends only on u, or is the aforementioned additive or multiplicative nonlinearity (or somewhat more general). When the nonlinearity is superlinear and of source-type, one cannot expect the boundary datum to be arbitrarily large. Even for the simpler model

$$-\Delta u = u^2 \qquad \text{in } \Omega \subset \mathbb{R}^2,$$

solutions exist if and only if the boundary trace satisfies $\|\nu\|_{\mathfrak{M}(\partial\Omega)} \leq \rho^*$ for some extremal parameter ρ^*, as initially proved by Bidaut-Véron and Vivier. NGuyen's sharpest result in this direction concerns the equation

$$-L_\mu u = u^p$$

He proves that given a boundary condition of the form $\rho\nu$, where $\rho \in \mathbb{R}_+^*$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ has unit mass, a solution exists in the subcritical regime if and only if $\rho \leq \rho^*$ for some extremal parameter ρ^*, while in the supercritical case the problem is never solvable if ν is a Dirac mass δ_y, $y \in \partial\Omega$. With Gkikas, he considered the more general equation $-L_\mu u = u^p + \sigma\tau$, $\sigma \in \mathbb{R}_+^*$, in the interesting situation where τ is a positive *unbounded* measure which is integrable only against a fractional power δ^α of the distance to the boundary, thereby extending results of Bidaut-Véron and Yarur to the case of the Hardy potential. In such circumstances, a necessary and sufficient condition for existence can be derived. in particular, admissible boundary traces must be absolutely continuous with respect to a capacity that depends on p *and* α. Finally, together with Gkikas, he opens up the exploration of the gradient-dependent case (for the model additive and multiplicative nonlinearities). They show existence for small data in the subcritical regime and sufficient conditions (expressed in terms of capacities again) in the supercritical case. Even in the classical case $\mu = 0$, the results seem new.

I hope that I have conveyed with these lines the idea that the research interests of NGuyen form a rich, difficult and fascinating topic. Reading his manuscript has convinced me that he is now a leading expert in the nonlinear analytic and potential theoretic tools used in this field and that he has obviously achieved the maturity to become a PhD advisor on these subjects. I therefore recommend that he defends his thesis in the warmest terms. It is also nice to see the fruitful collaboration that he has started with Gkikas: Marcus and Véron must be satisfied that their students are developing further a subject in which they have put so much efforts. As a final word, I would like to encourage NGuyen and his collaborators to go beyond the analytic aspects and try to revive the probabilistic side: the Brownian snake and superprocesses can be used for general elliptic equations (with smoothly bounded coefficients) of the type considered here, so it should be really interesting to see if the theory can further be extended to the case of the Hardy potential. The geometric aspects, in particular the singular Yamabe problem, is in my opinion another fascinating direction of outreach for this line of research.